Numerical computation of cable curves for cable railways

M. Wenin, CPE, Weingartnerstr. 28, 39011 Lana, BZ, Italy *
E. Oberhuber, imo, Weingartnerstr. 42, 39033 Algund, BZ, Italy†

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Abstract

We present an efficient numerical algorithm for the computation of curves for elastic cables of cable railways. The cable load is distributed over an arbitrary number of supports. The cable is assumed to move along the supports without friction and is held fixed at the endpoints of the line. The weight of the car is modelled as a point load at arbitrary position along the cable car line. For given temperature we minimize the total energy of this system numerically to determine the equilibrium positions of both the cable and the car. The resulting cable curve is exact up to first order in the strain. Numerical results demonstrate good performance for the computation of quasi-static cable movements. In a numerical example we demonstrate the graphical output of a MATLAB® program.

Keywords: Cable Railway, Cable spans, Elasticity, Numerical Computation, Minimization of Energy

1 Introduction

The problem of finding the equilibrium position of an inextensible cable and its solution is not new (catenary: Leibniz, Huygens, Bernoulli, 1691 [Routh(1891)]). The same holds true for an elastic cable also [Schell(1880)]. We have been unable to find out who solved this problem for the first time, but it was not G. Galilei, who assumed that the catenary is a parabola (probably the highly strained catenary only, the history in this case seems to be quite involved [Szabo(1987)]). In fact, in many applications the approximation of the exact cable curve in the form of a parabola is sufficient and has commonly been used up to now. It allows a quick estimation of the cable tension or the mid-span catenary sag and sufficient accuracy for technical application, in particular in the field of rope–ways, is reachable [Czitary(1962)]. The question arises why we one should look into such an problem again? The answer is simple. We want take advantage of

*corresponding author. e-mail: markus.wenin@cphysics.com
†e-mail: e.oberhuber@ing-mo.com

modern computer power and high level programming languages available and set up a consistent, closed set of equations to solve the core problems in the construction of cable cars exactly and fast: the determination of maximum cable tensions and mid–span catenary sags. Furthermore we are interested in bringing the information for the planning engineer to the computer screen graphically and in well organized form.
We consider an elastic cable (in the linear regime), distributed over an arbitrary number of supports fixed at the endpoints (we do not consider here the situation, where the cable is spanned with a movable weight at one end of the line). A point load (cable car) is sitting at an arbitrary position in the line. Our computational method is based on a minimization of the total energy of the system [Courant and Hilbert(1993)]. Such strategies are widely used in physics and technology (Ritz - variational principle [Schmutzer(1989), Rennert and Schmiedel(1995)], such as optimal control of quantum systems [Wenin, Roloff, and Pötz(2009)]) or in elasticity theory (principle of minimal free energy [Landau and Lifschitz(2010)]). This approach is ideal for static problems because one computes only the smallest number of quantities needed to characterize completely the physical system [Garcia, Carnicero, and Torres(2009)].
One can compare this situation with equilibrium thermodynamics and kinetic theory. In equilibrium a few parameters (e.g. pressure, volume) are sufficient to describe the complete system, whereas in non-equilibrium the determination of a distribution function is required. This argument immediately shows the advantage of our approach when compared with the usual discretization method, where the continuous cable curve is replaced by a large number of chain links. Obviously we cannot describe transversal oscillations of or longitudinal compression waves in cables (a "non-equilibrium" state of the cable). There is a rich scientific literature about this topic [Volmer(1999), Pataria(2008), Rega(2004), Starossek(1994)] [Gattulli, Martinelli, Perotti, and Vestroni(2004), Lacarbonara, Pacitti(2008)]. However the method is sufficient to compute all relevant numbers and data for the engineer to design a plant and check the safety required by law [CEN-Norm(2009)]. The program developed by St. Liedl, written in FORTRAN, present a remarkable piece of work but it does not take full advantage of graphical display of output available now [Liedl(1999)].
In this publication the attention is concentrated on the physics and the mathematical description of the problem. Technical aspects are regarded only as far as they are necessary.
The paper is organized as follows: in Sec. 2 we present the basic formulas used in the computations, e.g. the length of the cable. We discuss the energy of the complete system, define the dependent variables and the constraints of the minimization problem. Sec. 3 contains a numerical example for the optimization solution scheme. For now, we do not discuss problems, such as ice covered cables, wind loads, or friction resistance on the supports [Engel and Löscher(2003)]. We intend to do this in a further publication in the near future as discussed below.

2 Theory

In this section we give a brief overview of the main aspects of our approach and the most important formulas used in the computer program. The central point of the computation is the energy of the elastic catenary in a homogeneous gravitational field and its temperature dependence.

2.1 Basics

Let us start with some basic useful relations. The formulas in this section are used in several applications during the main computation and the determination of derived quantities (support pressure or various angles etc.) The supports themselves are considered to be point–like. All formulas are correct up to first order in the elastic strain. Because we consider two dimensional problems only, we use cartesian coordinates with abscissa x and ordinate y (see Fig. 1).

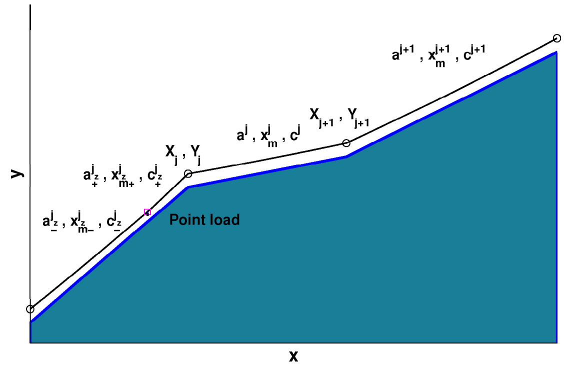

Figure 1: Sketch of the system under consideration and some definitions of quantities. The figure shows a cut of a cable railway, coordinates of support no. j are denoted as X_j, Y_j. The point load is sitting at an arbitrary position within the line, we assume the cable can move without friction on the supports. The elastic support cable for each span is characterized by three parameters.

2.1.1 "Elastic" catenary

The analytical expression for the "elastic catenary"[1] for one span is given by [Schell(1880)]

$$y(x) = y_0(x) + \delta y(x) \ , \qquad y_0(x) = a \cosh\left(\frac{x - x_m}{a}\right) - c \ . \tag{1}$$

Here $y_0(x)$ is the familiar catenary for the ideal flexible, inextensible cable. The cable parameters a, x_m and c uniquely define the position of the

[1] We use this terminus to distinguish it from the familiar catenary.

catenary in the plane. a is the radius, x_m the abscissa and $a - c$ the ordinate of the curvature at the vortex. The expression

$$\delta y(x) = -\frac{1}{2} k a^2 \sinh^2 \left(\frac{x - x_m}{a} \right) \quad , \qquad k \equiv \frac{g \rho_L}{\mathcal{E} A} \ , \tag{2}$$

accounts the deviation from it (g is the earth acceleration, $\mathcal{E}$ the isothermal modulus of elasticity and A the cross section of the cable, respectively). Eq. (1) is not exact but a series expansion, valid for small elastic strains (the exact expression is also known in parameter form). In practical situations the maximum strain $\Delta l / l$ is less than 0.5 %, so the linear theory is sufficient for our purpose. As one can see, for a given elasticity the catenary in any case is characterized by three parameters, the generalized coordinates, $\{a, x_m, c\}$. The linear mass density depends on temperature T, $\rho_L = \rho_L^0 [1 - \alpha_{\mathrm{L}}(T - T_{\mathrm{ref}})]$, where α_{L} is the coefficient of linear thermal expansion, but the influence of this effect on the numerical results is small compared to the thermal deviation of the cable length. ρ_L^0 is the mass density of the tensionless cable for the reference temperature T_{ref}. We assume no temperature gradients, so in practical situations one should use a mean temperature for both T and T_{ref}.

2.1.2 Length of the cable

To compute the length of the cable between two supports with coordinates (X_j, Y_j) and (X_{j+1}, Y_{j+1}), called l_j, we need two cable parameters a^j, x_m^j. Using the well known arc length integral $\int \sqrt{1 + y'(x)^2} dx$, we perform a series expansion of the integrand (the prime denotes the derivative respect to x), where we use Eq. (1) and Eq. (2). Omitting the details of the calculations, we arrive at (we assume always $X_{j+1} > X_j$),

$$l_j = G(a^j, x_m^j; X_{j+1}) - G(a^j, x_m^j; X_j) \ . \tag{3}$$

The length l_j is written as a difference of a function G, evaluated for two coordinates. G itself is defined as

$$G(a, x_m; x) := F_1(a, x_m; x) + \delta F_1(a, x_m; x) \ , \tag{4}$$

where the well known function for the inextensible cable is

$$F_1(a, x_m; x) := a \sinh \left(\frac{x - x_m}{a} \right) \ , \tag{5}$$

and

$$\delta F_1(a, x_m; x) := -\frac{1}{4} a k \left[-2x + a \sinh \left(\frac{2(x - x_m)}{a} \right) \right] \ . \tag{6}$$

takes into account the elastic strain of the cable. $G(a, x_m; x)$ does not depend on the parameter c.

2.1.3 Tension, elastic strain and lengthening of the cable

The computation of the tension and the elastic lengthening of the cable is an essential task for the computation of the cable curve. We present therefore here some important results, used later several times, when the length of the tensionless cable at a fixed temperature is prescribed at a given value. In general, the differential tension is $dF = \rho_L^{ten}(x) g dy$, where $\rho_L^{ten}(x)$ is the linear mass density of the stretched cable. This means, $\rho_L^{ten}(x)$ depends on the strain and is therefore a function of the position. Integrated one obtains, for small strain, the expression

$$F(x) = F_{\mathrm{V}} + \rho_L g[y(x) - Y_1] - \frac{\rho_L g}{A\mathcal{E}} \Big\{ F_{\mathrm{V}} + \frac{\rho_L g}{2}[y_0(x) - Y_1] \Big\} [y_0(x) - Y_1] \ . \quad (7)$$

Here $Y_1 = y(X_1)$ is the y-coordinate of the valley station (where the cable is anchored) and F_{V} the cable tension at the valley station. By insertion of the appropriate $y_0(x)$, $y(x)$, Eq. (7) is valid for the entire cable, independent of the position and number of supports. To calculate the space–dependent strain

$$\varepsilon(x) = \frac{F(x)}{A\mathcal{E}} \ , \quad (8)$$

up to first order, it is sufficient to use

$$F(x) = F_{\mathrm{V}} + \rho_L g[y_0(x) - Y_1] \ . \quad (9)$$

For the elastic lengthening of the cable of span no. j we use the general expression

$$\Delta l_j = \int_{X_j}^{X_{j+1}} \varepsilon(x) \sqrt{1 + y'(x)^2} dx = \frac{\langle F_j \rangle}{A\mathcal{E}} l_j^0 \ , \quad (10)$$

where $\langle F_j \rangle$ is the mean tension within span no. j. Using the functions given by Eq. (5) and Eq. (14), the elastic lengthening is given by

$$\Delta l_j = \frac{1}{A\mathcal{E}} \Big\{ [F_{\mathrm{V}} - \rho_L g Y_1] \Big[F_1(a^j, x_m^j; X_{j+1}) - F_1(a^j, x_m^j; X_j) \Big] + \rho_L g \Big[F_2(a^j, c^j, x_m^j; X_{j+1}) - F_2(a^j, c^j, x_m^j; X_j) \Big] \Big\} \ . \quad (11)$$

This expression is also useful to compute the mean cable tension via Eq. (10).

2.1.4 Definition of two auxiliary functions

To compute the energy of the elastic cable in the gravitational field according to Eq. (19) and Eq. (20), we need two auxiliary functions, which we prefer to define already here. The first one is given by

$$H(a, c, x_m; x) := \int y(x) \sqrt{1 + y'(x)^2} dx \ . \quad (12)$$

The treatment of this integral up to first order in the strain is a straightforward but lengthy calculation. It is again advantageous to distinguish between the inextensible cable and corrections to it. So we set

$$H(a, c, x_m; x) = F_2(a, c, x_m; x) + \delta F_2(a, c, x_m; x) \ . \quad (13)$$

Here as before, the auxiliary function $F_2(a, c, x_m; x)$ arise from the computation of Eq. (12) with $y_0(x)$ from Eq. (1). The second auxiliary function $\delta F_2(a, c, x_m; x)$ contains the corrections due to elasticity. Explicitly we obtain,

$$F_2(a, c, x_m; x) = \frac{a}{4}\left[2x - 4c\sinh\left(\frac{x - x_m}{a}\right) + a\sinh\left(\frac{2(x - x_m)}{a}\right)\right] \ , \tag{14}$$

$$\delta F_2(a, c, x_m; x) = \frac{1}{4}ak\left\{ - 2cx - 2a^2\sinh^3\left(\frac{x - x_m}{a}\right) + ac\sinh\left(\frac{2(x - x_m)}{a}\right)\right\} \ . \tag{15}$$

The second function is defined as follows

$$B(a, c, x_m; x) := \int y_0(x)^2\sqrt{1 + y_0'(x)^2}dx \ . \tag{16}$$

We carry out the integral and obtain

$$B(a, c, x_m; x) = ac(x - x_m) + \frac{1}{4}(3a^3 + 4ac^2)\sinh\left(\frac{x - x_m}{a}\right) + \frac{a^3}{12}\sinh\left(\frac{3(x - x_m)}{a}\right) - \frac{a^2c}{2}\sinh\left(\frac{2(x - x_m)}{a}\right) \ . \tag{17}$$

2.2 Energy

The central quantity for our numerical strategy is the energy E of the physical system. We consider isothermal processes, therefore, more rigorously, we work with the free energy of the system in an homogeneous external field [Landau and Lifschitz(2010)]. The total energy consists of three parts, the potential energy of the cable E_{pot}, the elastic potential (strain) of the cable E_{str}, and the potential energy of the cable car (point mass) E_Z,

$$E_{\text{tot}} = E_{\text{pot}} + E_{\text{str}} + E_Z = E_{\text{cab}} + E_Z \ . \tag{18}$$

2.2.1 Computation of E_{pot} and E_{str}

The potential energy E_{pot} of a free span within two supports with x-coordinates X_j and X_{j+1} is given by

$$E_{\text{pot}}(a^j, x_m^j, c^j) = g\int_{X_j}^{X_{j+1}} \tilde{\rho}_L(x)y(x)\sqrt{1 + y'(x)^2}dx \ , \tag{19}$$

where $\tilde{\rho}_L(x) = \rho_L/[1 + \varepsilon(x)] \approx \rho_L[1 - \varepsilon(x)]$ is the linear mass density depending on x. Next we consider E_{str}. The general expression contains the material parameter $\mathcal{E}$ and the cross section A of the cable,

$$E_{\text{str}} = \frac{1}{2\mathcal{E}A}\int_{X_j}^{X_{j+1}} F(x)^2\sqrt{1 + y_0'(x)^2}dx \ . \tag{20}$$

We evaluate both integrals to obtain for the energy of the cable,

$$E_{\text{cab}}(a^j, x_m^j, c^j; X_j, X_{j+1}) = \rho_L\ g\ \Big[H(a^j, c^j, x_m^j, X_{j+1}) - H(a^j, c^j, x_m^j, X_j)\Big] - \frac{(\rho_L g)^2}{2\mathcal{E}A}\Big[B(a^j, c^j, x_m^j, X_{j+1}) - B(a^j, c^j, x_m^j, X_j)\Big] + \frac{(F_{\text{V}} - \rho_L g Y_1)^2}{2\mathcal{E}A}\Big[F_1(a^j, x_m^j, X_{j+1}) - F_1(a^j, x_m^j, X_j)\Big]\ . \tag{21}$$

We remark that E_{cab} depends on temperature via ρ_L and, depending on the constraints, also via F_{V}.

2.2.2 E_{tot} for empty spans

For N empty spans we have a function of $3 \times N$ variables,

$$E_{\text{tot}}(a^1, x_m^1, c^1; a^2, x_m^2, c^2; \dots a^N, x_m^N, c^N) = \sum_{j=1}^{N} E_{\text{cab}}(a^j, x_m^j, c^j; X_j, X_{j+1})\ . \tag{22}$$

One can see that E_{tot} here is the sum of all single span energies.

2.2.3 E_{tot}^Z for the span carrying the point load

Let us assume the point load with mass m_{Z} is placed within span no. j_z, as indicated in Fig. 1. X_{Z} is its $x-$ coordinate, $X_j \le X_{\text{Z}} \le X_{j+1}$. Due to the presence of the point load this span is divided into two spans, characterized by two sets of parameters $\{a_-^{j_z}, x_{m-}^{j_z}, c_-^{j_z}\}$ and $\{a_+^{j_z}, x_{m+}^{j_z}, c_+^{j_z}\}$. The $-$ sign denotes the left, the $+$ sign the right span. The energy of the charged span is therefore a sum of three parts, two for two empty spans and one for the energy of the point load, $F_\perp y(X_{\text{Z}})$,

$$E_{\text{tot}}^Z = E_{\text{cab}}(a_-^{j_z}, x_{m-}^{j_z}, c_-^{j_z}, X_{j_z}, X_{\text{Z}}) + E_{\text{cab}}(a_+^{j_z}, x_{m+}^{j_z}, c_+^{j_z}, X_z, X_{j_z+1}) + F_\perp y(X_z)\ . \tag{23}$$

The coordinates $\{X_{\text{Z}}, y(X_{\text{Z}})\}$ are either computed with the left or the right parameter set in Eq. (1). The value of $F_\perp$ is not equal to the weight of the car, but reduced due to the vertical force component of the hauling cable. Assuming an action angle $\psi = (\alpha_+ + \alpha_-)/2$ of the upper and lower hauling cables (see Fig. 2), we can write for the effective support cable load (this assumption requires a high tension of the hauling cable, usually realized in practical situations [Czitary(1962)])

$$F_\perp = m_{\text{Z}} g \cos^2(\psi)\ . \tag{24}$$

We express the angle ψ in terms of $\{a_+^{j_z}, x_{m+}^{j_z}\}$ and $\{a_-^{j_z}, x_{m-}^{j_z}\}$ to describe the energy of the point load self–consistently (for an investigation of dynamical cable-point load interactions see [Bryja, and Knawa(2011)]).

2.3 Constraints

Apart from the definition of the energy we have to identify the constraints in order to get a solution. All constraints are treated using Lagrange

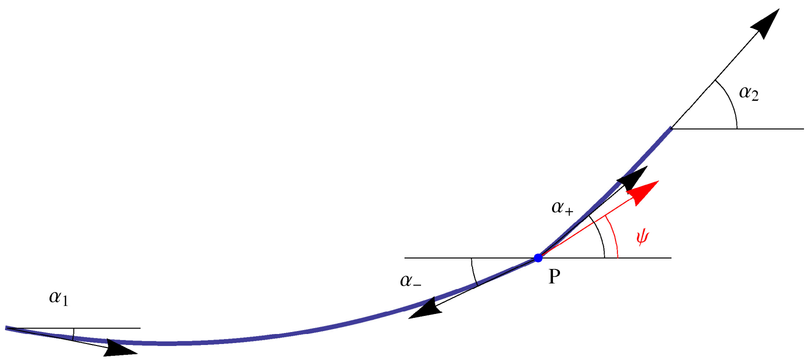

Figure 2: Some definitions of angles. The car (point load) is at the position P. The (upper and lower) hauling cables act under the angle $\psi = (\alpha_+ + \alpha_-)/2$, where α_-, α_+ are the angles of the tangents at P. The angles α_1, α_2 are derived from the cable parameters.

multipliers. The constraints are: continuity of the cable curve at the supports and the point load position. The length of the tensionless cable for a given temperature (conservation of mass, formulated with Eq. (3) and Eq. (11)). To increase numerical stability we use further the condition that the horizontal cable force is constant within one span.

2.3.1 Empty cable at T_{ref}

The computation starts with the empty cable at a given reference temperature T_{ref} and cable tension F_V. The linear mass density ρ_L^0 corresponds to T_{ref} and zero tension. The solution is completely determined by the constraints. One minimizes a constant function augmented by the constraints: sum of horizontal forces in a span is zero, and the continuity of the cable line at the supports.

2.4 Minimization

We perform the minimization of E_{tot} by numerical methods available in the MATLAB® optimization toolbox,

$$\{a^j, x_m^j, c^j\}_{\text{eq}} = \min_{\{a^j, x_m^j, c^j\}} \{E_{\text{tot}} + E_{\text{tot}}^z\} \ , \tag{25}$$

subject to the constraints. The solution $\{a^j, x_m^j, c^j\}_{\text{eq}}$ represents the set of equilibrium parameters which minimizes the energy. The routine uses local, gradient based minimization methods, and an initial guess is required.

2.4.1 Starting values for the empty cable

We assume a given cable tension at the valley $F_{\rm V}$, support coordinates and linear mass density ρ_L^0 for $T_{\rm ref}$. The derivation of the following starting values for the cable parameters is based essentially on the parabola approximation of the cable line (see Eq. (32)). Using the auxiliary quantities: $\Delta X_j = X_{j+1} - X_j$, $\Delta Y_j = Y_{j+1} - Y_j$, $\Delta S_j = \sqrt{\Delta X_j^2 + \Delta Y_j^2}$, and $\zeta_j = \frac{l_c + Y_{j+1} - Y_1}{l_c + Y_j - Y_1}$, $w_j = \sqrt{4\zeta_j^2 \Delta Y_j^2 - (1-\zeta_j^2)^2 \Delta X_j^2}$, where $l_c = F_{\rm V}/\rho_L^0 g$ is a characteristic length, we obtain:

- case $Y_{j+1} < Y_j$

$$(a^j)_0 = \frac{\Delta X_j^2 [\Delta Y_j (1+\zeta_j^2) - w_j]}{2\Delta S_j^2 (\zeta_j^2 - 1)} \ , \tag{26}$$

$$(x_m^j)_0 = \frac{X_{j+1}^2 - X_j^2 - 2a^j \Delta Y_j}{2\Delta X_j} \ . \tag{27}$$

- case $Y_{j+1} = Y_j$ $(\zeta_j = 1)$

$$(a^j)_0 = l_c + Y_j - Y_1 \ , \tag{28}$$

$$(x_m^j)_0 = \frac{X_j + X_{j+1}}{2} \ . \tag{29}$$

- case $Y_{j+1} > Y_j$

$$(a^j)_0 = \frac{\Delta X_j^2 [\Delta Y_j (1+\zeta_j^2) + w_j]}{2\Delta S_j^2 (\zeta_j^2 - 1)} \ , \tag{30}$$

  $(x_m^j)_0$ the same as for the case $Y_{j+1} < Y_j$.

One is left with the expression for $(c^j)_0$ which, in any of the three cases, is given by

$$(c^j)_0 = \frac{\Delta X_j^4 + 4(a^j)_0(2\Delta X_j^2 + \Delta Y_j^2) - 4(a^j)_0 \Delta X_j^2 (Y_j + Y_{j+1})}{8(a^j)_0 \Delta X_j^2} \ . \tag{31}$$

2.5 Minimal value for the cable tension

In which case does a solution for the parameters $\{a, c, x_m\}$ exist if we consider an inextensible cable with given length between two points? The answer to this question is quite simple: when the length of the cable is larger than the euclidian distance of the points. The same question occurs if the cable tension $F_{\rm V}$ instead of its length is prescribed. This is the case in practice, where $F_{\rm V}$ is set by hand and is of the order $\approx 10\times$ the weight of the car [Czitary(1962)]. To clear the ambiguities in this case we consider one span only. Given the tension of the cable $F_{\rm V}$ at one end (let's say the valley). If we use the model of the inextensible cable it is possible to derive the following exact relation between $F_{\rm V}$ and the span parameter a

(the other two parameters are eliminated using the balance of forces and the geometric constraint),

$$F_{\rm V} = \frac{1}{2}\rho_L g \Bigg[-(Y_2 - Y_1) + \coth\left(\frac{X_2 - X_1}{2a}\right) \times \\ \sqrt{(Y_2 - Y_1)^2 + 4a^2 \sinh^2\left(\frac{X_2 - X_1}{2a}\right)}\Bigg] . \qquad (32)$$

We assume now that the coordinates of the endpoints X_1, Y_1 and X_2, Y_2 and ρ_L are given. Then we can plot $F_{\rm V}$ as a function of the parameter a (see Fig. 3). We can see that $F_{\rm V}$ has a minimum $F_{\rm min}$. For $F_{\rm V} < F_{\rm min}$ no solution exists. For $F_{\rm V} > F_{\rm min}$ two different cable parameters lead to the same cable tension. In practical situations for cable railways only the larger value a is relevant.
We compare an elastic with an inextensible cable. Both cables should have the same length and linear mass density, when no tension is applied. Physically it is clear that the tension F_V (and as a consequence $F_{\rm min}$) for the elastic cable is smaller as for the inextensible cable. We conclude: if $F_{\rm V} > F_{\rm min}$, evaluated for the inextensible cable, is true, a solution exists for the elastic cable also.

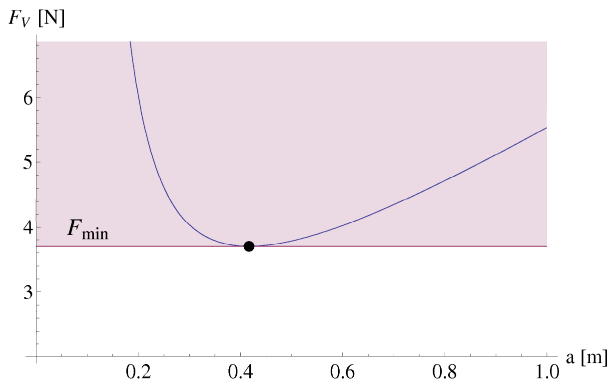

Figure 3: Example with $\rho_L = 0.5$ kg/m and $Y_2 - Y_1 = 0$, $X_2 - X_1 = 1$ m gives a minimal value $F_{\rm min} = 3.7$ N.

3 Numerical example

In this section we present a numerical example which serves to describe the structure of the algorithm. The input data correspond to a plant in Southern Tirol, Italy. This private cable railway leads from altitude 1696 m to 2142 m and conveys at the most four people. The support cable is a locked coil rope of diameter 21 mm. There are 5 supports (without the endpoints) and $N = 6$ spans.

3.1 Structure of the algorithm

A practical computation starts with the empty cable and the following input parameters[2]

- parameters for the (tensionless) cable: linear density, elasticity, cross section, coefficient for thermal expansion α_{L},
- coordinates of the supports,
- reference temperature T_{ref},

In a first run we compute from this data the empty cable curve and the length $l_0 = \sum_{j=1}^{N} l_j^0$ of the cable without tension for the temperature T_{ref}. In the following are computed other relevant quantities, such as support pressures or various angles.
We continue with the computation of the free cable at a given temperature T. The length of the tensionless cable l_T is given by

$$l_T = l_0[1 + \alpha_{\mathrm{L}}(T - T_{\mathrm{ref}})] \ . \tag{33}$$

This value is used as a constraint for the second run of the computation. We use the parameter set $\{a^j, x_m^j, c^j\}_{\mathrm{eq}}$ of the first result(temperature T_{ref}) as starting values for the new run. The expression for the elastic lengthening Eq. (11) contains F_{V}. We express F_{V} by the weight of the cable ($= l_0 g \rho_L^0$) using the balance of the forces.
When the car is within the line (at a temperature T), F_{V} is not constant, but depends on the position of the load (in contrast to the situation, where the cable is stretched by a weight). We compute it again by a balance of forces.

3.2 Numerical results

We use the following cable parameters: $\rho_L^0 = 2.53$ kg/m, $A = 301$ mm^2, $\mathcal{E} = 160000$ N/mm^2, $\alpha_{\mathrm{L}} = 11.7 \times 10^{-6}$ K^{-1}. The tension in the valley station is $F_{\mathrm{V}} = 1.079 \times 10^5$ N and the reference temperature $T_{\mathrm{ref}} = 20$ °C respectively. The maximum mass of the vehicle (empty car plus four passenger) is $m_{\mathrm{Z}} = 1080$ kg and we use this value for the computations presented here. In Tab. 1 we show the support coordinates and tangent angles of the cable at the supports. The latter are computed by elementary calculus and important quantities to design the supports. The support pressures F_n are listed also, whereby for the valley– and mountain stations F_n per definition points in the cable tangent direction, i.e. is identical with the cable tension at this position.
Tab. 2 shows the starting and final values of the span parameters $\{a^j, x_m^j, c^j\}$ of the empty cable at the reference temperature in units of $l_c = 4347.42$ m. As one can see, the starting values are already near the exact results.
The computation of the length of the tensionless cable at T_{ref} gives 1831.26 m, and for the elastic lengthening we obtain 4.38 m, whereas the sum of the chords is 1835.21 m. One can check these results immediately using Eq. (10), extended to all spans, which gives for a mean cable tension of

[2] The structure of the algorithm is widely determined by practical guidelines. See also, Eng. Vittoriano Vitali: "Cableway Design Package (C.D.P.)".

≈ 110 kN (see Fig. 5) a lengthening of ≈ 4.2 m. Further results for the empty cable are obtained for different temperatures and discussed in the next section.
The main part of the computations deals with the situation, where the point load is present. To simulate the quasi–static movement of the car, we compute the equilibrium cable configuration for a series of car positions, starting at the valley station (for a study with one span only and a moving mass see [Wang, Rega(2010)]). For the presented example we have choosen 120 equidistant positions, at a distance of 14.63 m. In order to avoid numerical problems, we ensure a sufficient distance to the supports.

3.2.1 Discussion of the figures

As the most part of the numerical results are presented in figures, we discuss these separately. If the contour of the mountain is available, we can compute and visualize important quantities, which are immediately valuable for the engineer.
We start with Fig. 4, which gives an overview of the plant. The positions of the supports and cable lines for the empty cable are shown. Three temperatures $T_{\rm ref}$, $T_{\rm min} = -30$ °C, and $T_{\rm max} = 30$ °C are considered. The bar graph shows the mid–span catenary sag D of the empty cable for all spans and three temperatures. Because D is measured from the chord, it is also computable if the mountain contour is missing.
In Fig. 5 we have plotted the cable tension as a function of position for three temperatures, and the strain, always for the empty cable. Self–evidently the tension increases for increasing altitude, as Eq. (7) shows. The mean cable tension for the different spans is an important quantity when computing the constraint "constant cable length without tension", therefore we have plotted these values also.
Fig. 6 shows probably one of the most important results of the simulations, namely the distance from ground of the car, when moving along the line. We remember the reader that we have neglected the friction resistance of the cable on the supports. Obviously this simplification changes the result, but an exact determination of the state of the system requires an integration of the equations of motion for the cable and the point load. The solution depends on the initial conditions, such that we are far away from a simple model with a unique solution. Therefore we are convinced that Fig. 6 contains valuable information for the planning phase of a cable railway and numbers that are representative. In Fig. 7 we have plotted the support pressure for all supports and positions of the car. As one can see, all curves are in the negative region, which means that the cable in any case (for all point load positions $X_{\rm Z}$) does not leave the support. These results are important to decide whether an additional securing of the cable is necessary or not. Fig. 8 shows an analysis of the cable tension during the quasi-static movement of the car. The density plot gives the complete information about the cable tension, i.e. the tension of the cable at all positions x, for all positions of the point load X_Z (again: without considerations of friction resistances. A simple cumulant sum of the support friction resistances for fixed cables is doubtful, and the norm

does not really help in this case [CEN-Norm(2009)]). The presented plot is for the minimal temperature, which leads to the maximum tension. The search of the maximum gives for $x = 1255.2$ m (fifth support) and $X_Z \approx 590$ m (middle of the second span) a value of 158.8 kN. Fig. 9 presents a summary of subordinate data: the catenary sag of the cable for $T_{\min}$, T_{ref}, $T_{\max}$, when the car is in the middle of the span. The second bar graph shows more technical information, namely the relative elastic lengthening of the cable, where the elastic lengthening due to the proper weight was subtracted. Fig. 10 shows the displacement of the support cable on the supports as a function of the car–position. Here the assumption of vanishing cable–support friction resistance is essential.

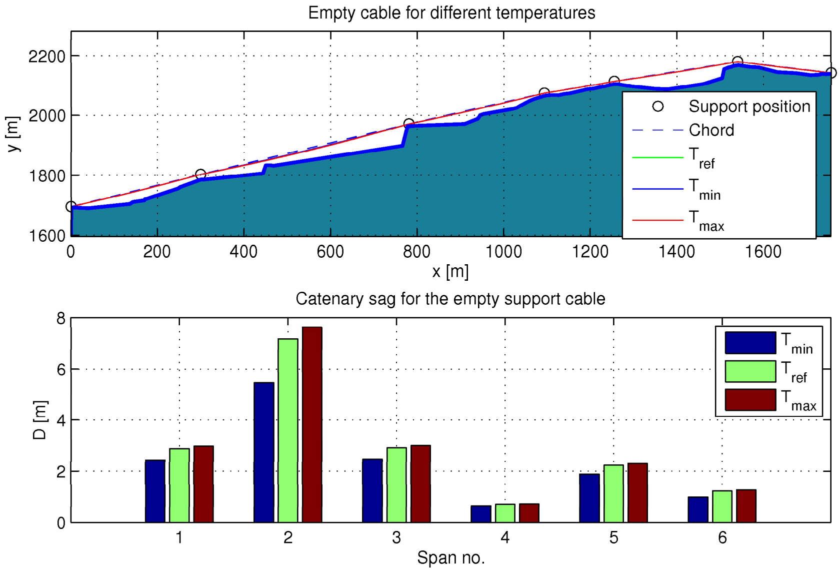

Figure 4: Sketch of the plant and cable lines of the empty cable. The lower part shows the mid–span catenary sag for different temperatures. D is defined as the maximum vertical distance of the cable from the chord.

4 Conclusions

The presented work discusses a numerical method to compute the equilibrium state of an elastic cable, distributed over an arbitrary number of supports and charged with a point load at an arbitrary position. We use the model of the "elastic catenary" and minimize the total energy of the system, consisting of the elastic cable and the point load in a gravitational field. The mathematical expressions to compute the energy are presented. Because a local, gradient–based minimization algorithm is used, we give

Table 1: Support coordinates, cable tangent angles and pressures of the empty cable on the supports for the reference temperature (* values are identical to the cable tension).

support i	1 (vall.)	2	3	4	5	6	7 (mount.)
X_i [m]	0.0	299.8	781.2	1094.2	1255.2	1540.8	1756.0
Y_i [m]	1696.0	1803.0	1971.5	2074.5	2114.3	2180.0	2142.5
α_1^i [°]	17.67	16.16	16.36	12.91	11.24	-11.16	-
α_2^i [°]	-	21.58	22.24	20.20	14.85	14.65	-8.60
F_n [kN]	107.9*	10.35	12.04	14.75	7.46	53.57	118.98*

Table 2: Starting and final values for the parameters $\{a^j, x_m^j, c^j\}$ for all spans. The starting values, denoted with brackets and subscript 0, are computed using the equations from Sect. 2.4.

i	$(a^i)_0$	$(x_m^i)_0$	$(c^i)_0$	a^i	x_m^i	c^i
1	0.8969	-0.2856	0.5522	0.9506	-0.2986	0.6077
2	0.9280	-0.1995	0.5521	0.9818	-0.2123	0.6076
3	0.9681	-0.1044	0.5564	1.0177	-0.1157	0.6074
4	1.0285	0.0160	0.5783	1.0570	0.0109	0.6073
5	1.0474	0.0806	0.5817	1.0725	0.0764	0.6072
6	1.0739	0.5663	0.5933	1.0876	0.5682	0.6072

formulas to compute the starting values for the solver for the empty cable. We discuss the influence of the temperature on the energy and the constraints. In our new program, realized in MATLAB® , we compute the equilibrium cable configuration for a large number of positions of the car. Within a loop we displace the position of the point load, starting from the valley station up to the mountain station, to obtain all relevant information (cable tension, support pressures, various angles etc.) for all positions of the car. In a numerical example of a plant realized in Southern Tirol, Italy, we demonstrate our theoretical approach. To give an overview of the main results, the data are presented almost as graphics, welcome to the engineer.

In this work we do not discuss the important questions about the friction resistance of the cable on the supports, and also many other technical aspects (ice covered cable, wind [Impollonia, Ricciardi, and Saitta(2011)]) are ignored at the moment. We plan to do this, on the basis of the presented results in the next time. As a further ambitious goal we intend to test optimization algorithms as auxiliary instruments to find out the optimal cable type or the optimal support positions. The possibilities in this research direction are numerous [Kazakoff(2012)].

5 Acknowledgement

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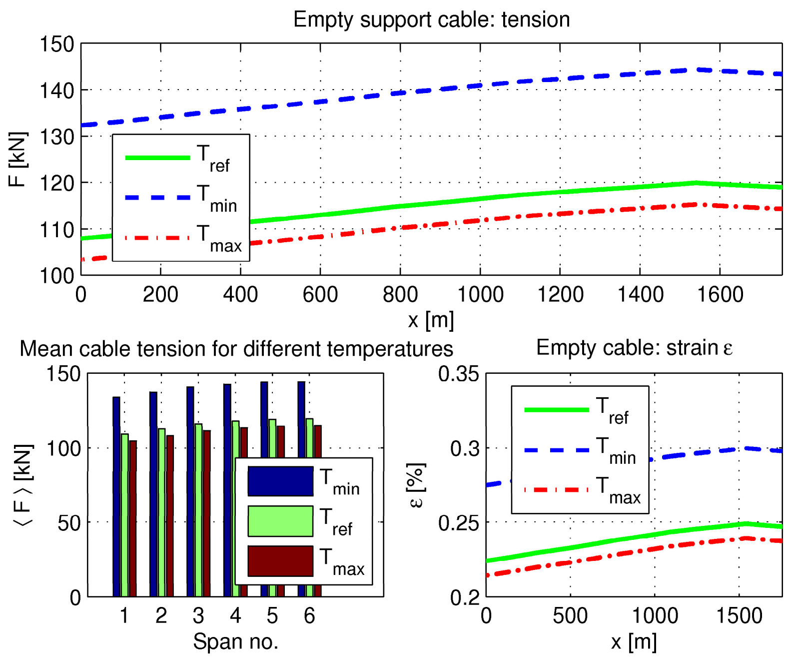

Figure 5: The cable tension for different temperatures $T_{\min}$ and $T_{\max}$ as a function of x. The bars show the mean tensions for the spans, the strain is computed according to Eq. (8).

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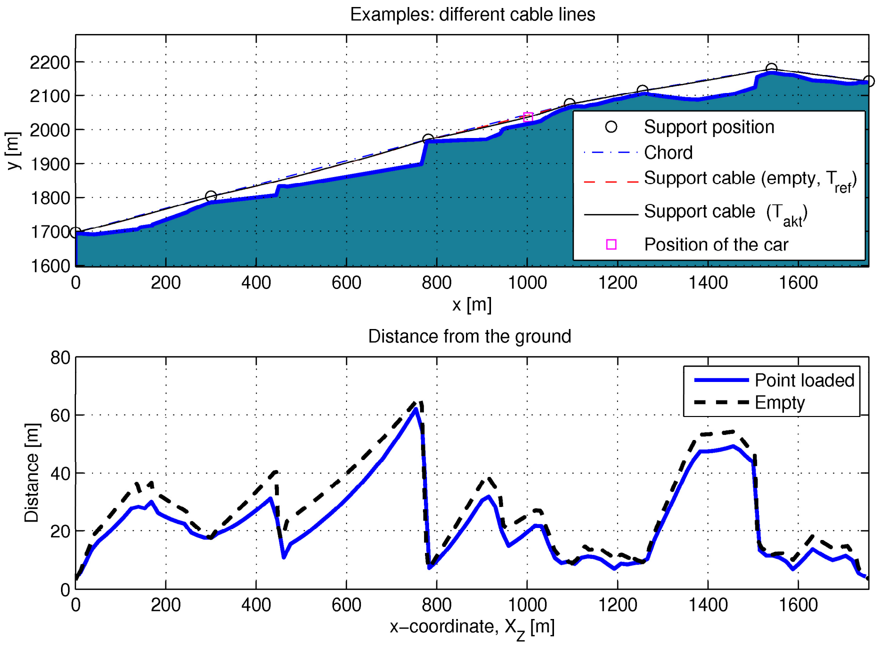

Figure 6: Sketch of the system showing the mountain contour, the support positions and several cable lines. The temperature–dependent distances "car–ground" and "empty cable–ground" are immediately available. Here we have choosen $T = T_{\text{ref}}$.

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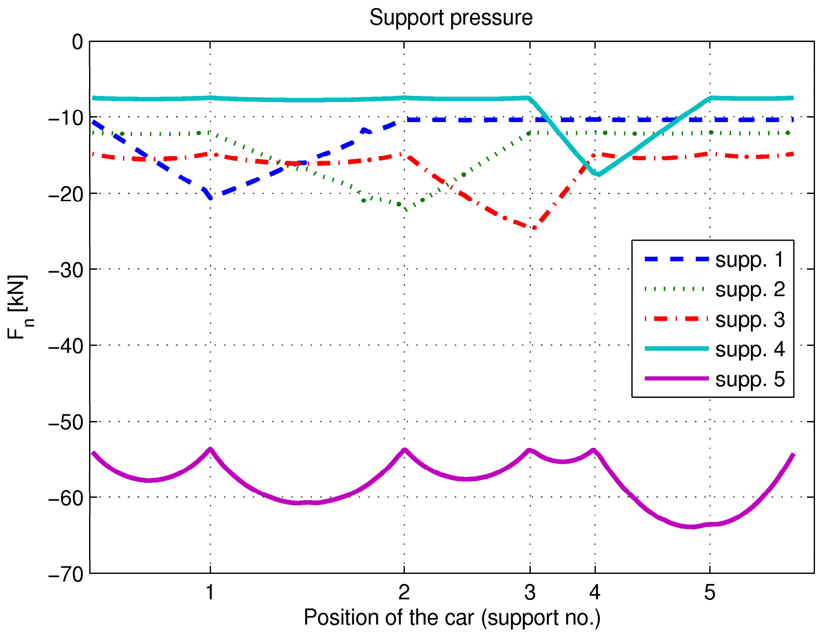

Figure 7: Support pressure F_n for all supports and all positions of the car. F_n is the force perpendicular to the half cable deviation angle. As one can see, all lines are in the negative region, so it is guaranteed that the cable does not leave the support. The plot corresponds to the temperature T_{ref}.

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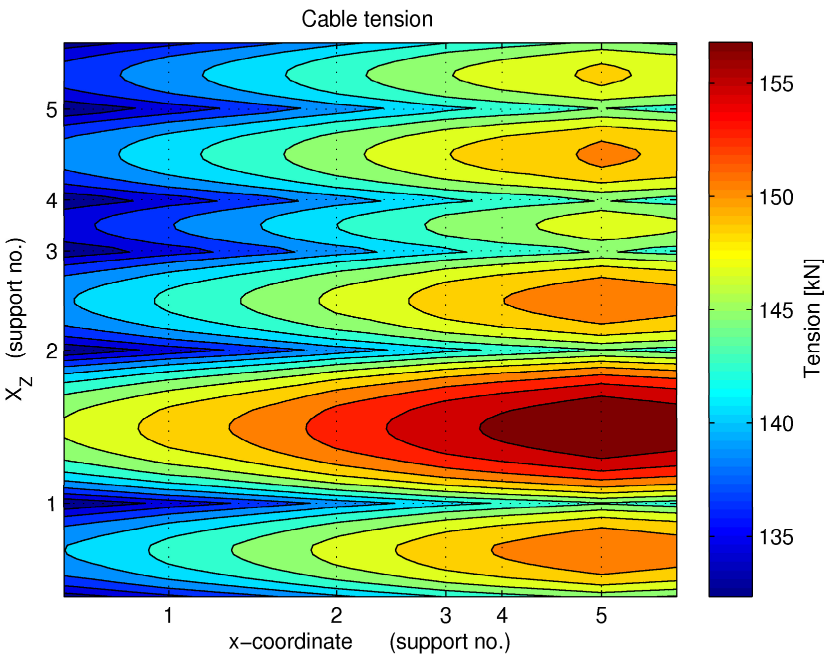

Figure 8: Cable tension as a function of position x and car position X_Z. Here the temperature is set $T_{\min} = -30$ °C, which gives the maximum tension.

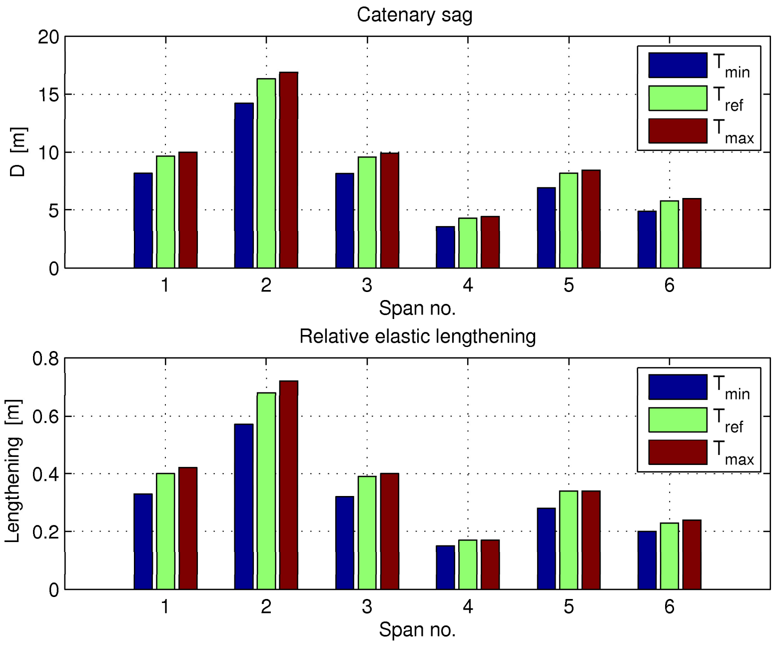

Figure 9: Mid-span catenary sag, if the car is in the middle of the span, for the three temperatures. The lower panel shows the relative elastic lengthening of the entire cable for this case. The indicated lengthening is caused by the point load. Additional we have for the empty cable the elastic lengthening due to the proper weight, which numerically gives 5.29/4.38/4.21 m for $T_{\rm min}/T_{\rm ref}/T_{\rm max}$, respectively.

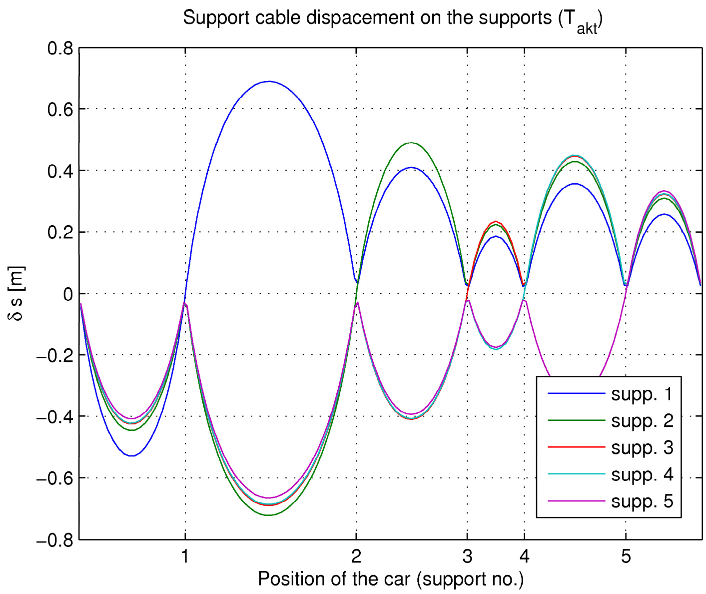

Figure 10: Displacement of the support cable δs during the ride of the point load. $\delta s < 0$ means the cable moves to the left.

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